Int. J. Solids Structures, 1975, Vol. 11, pp. 569–573. Pergamon Press. Printed in Great Britain

ON TRANSVERSE BENDING OF PLATES, INCLUDING THE EFFECT OF TRANSVERSE SHEAR DEFORMATION†

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(*Received* 8 *July* 1974)

In this paper we return once more to the problem of transverse bending of homogeneous transversely isotropic plates, including the effect of transverse shear deformation. Our object is to describe an approach to the problem which reproduces our earlier equations of *two*-dimensional plate theory[2] exactly, while at the same time leading to new supplementary information concerning certain practically important *three*-dimensional aspects of the problem. The significance of this supplementary information will be shown for the example of the problem of pure bending of an infinite plate with a circular hole.

Our starting point for the derivation of two-dimensional plate theory is once more the conventional system of equilibrium equations of three-dimensional theory

$$\sigma_{x,x} + \tau_{yx,y} + \tau_{zx,z} = 0, \text{ etc.} \tag{1}$$

together with stress strain relations which are here taken in the form

$$u_{,x} = \frac{\sigma_x - \nu\sigma_y}{E} - \nu_z\frac{\sigma_z}{E_z}, \quad v_{,y} = \frac{\sigma_y - \nu\sigma_x}{E} - \nu_z\frac{\sigma_z}{E_z}, \quad \frac{u_{,y} + v_{,x}}{2(1+\nu)} = \frac{\tau_{xy}}{E} \tag{2}$$

and

$$u_{,z} + w_{,x} = \frac{\tau_{xz}}{G}, \quad v_{,z} + w_{,y} = \frac{\tau_{yz}}{G}, \quad w_{,z} = \frac{\sigma_z}{E_z} - \nu_z\frac{\sigma_x + \sigma_y}{E_z}, \tag{3}$$

and together with face boundary conditions

$$z = \pm\tfrac{1}{2}h; \quad \sigma_z = 0, \quad \tau_{zx} = 0, \quad \tau_{zy} = 0, \tag{4}$$

where for simplicity's sake, and with no significant effect on the conclusions to be established, we omit consideration of loads distributed over the faces of the plate.

In our earlier work[2] we obtained a system of equations of two-dimensional theory upon introducing approximate stress distributions

$$\sigma_x = \frac{M_x}{h^2/6}\frac{z}{h/2}, \quad \tau_{xz} = \frac{V_x}{2h/3}\left[1 - \left(\frac{z}{h/2}\right)^2\right], \text{ etc.} \tag{5}$$

†Prepared with the support of the Office of Naval Research and the National Science Foundation. The author wishes to thank Gary E. Lee for assistance in obtaining the results presented in this paper.

into a suitable minimum complementary energy condition, and upon evaluating this condition in conjunction with an application of the Lagrange multiplier concept.

In what follows we retain our assumption that the distribution of transverse shears may be taken to be parabolic, approximately. However, we abandon our assumption that σ_x, σ_y and τ_{xy} vary linearly across the thickness (which is an entirely adequate approximation for the derivation of our approximate two-dimensional theory through use of the minimum complementary energy condition, without however being equally adequate for the determination of the effect of transverse shearing strains on the dependence of σ_x, σ_y and τ_{xy} on the thickness coordinate z). Instead we proceed as follows. We take account of the fact that, by equilibrium considerations, the transverse normal stress σ_z is small compared to the transverse shearing stresses τ_{xz} and τ_{yz}, and these in turn are small compared to σ_x, σ_y and τ_{xy}, by omitting σ_z altogether in the stress strain relations and by approximating τ_{xy} and τ_{yz} in these relations as follows

$$u_{,z} + w_{,x} = \frac{3V_x}{2hG}\left(1 - 4\frac{z^2}{h^2}\right), \quad v_{,z} + w_{,x} = \frac{3V_y}{2hG}\left(1 - 4\frac{z^2}{h^2}\right) \tag{6}$$

$$w_{,z} = -\nu_z \frac{\sigma_x + \sigma_y}{E_z} \tag{7}$$

and

$$\frac{\sigma_x}{E} = \frac{u_{,x} + \nu v_{,y}}{1 - \nu^2}, \quad \frac{\sigma_y}{E} = \frac{v_{,y} + \nu u_{,x}}{1 - \nu^2}, \quad \frac{\tau_{xy}}{E} = \frac{u_{,y} + v_{,x}}{2(1 + \nu)}. \tag{8}$$

We now integrate (7) in the form

$$w = W(x, y) - \int_0^z \nu_z \frac{\sigma_x + \sigma_y}{E_z} \mathrm{d}z \tag{9}$$

and introduce this relation into the approximate transverse shear stress strain equations (6). From these we obtain

$$u_{,z} = -W_{,x} - \int_0^z \left(\nu_z \frac{\sigma_x + \sigma_y}{E_z}\right)_{,x} \mathrm{d}z + \frac{3V_x}{2hG}\left(1 - 4\frac{z^2}{h^2}\right) \tag{10}$$

with a corresponding equation for $v_{,z}$. We next observe that in *these* equations the stress sum $\sigma_x + \sigma_y$ contributes an effect of the same order of magnitude as the stresses τ_{xz} and τ_{yz}. Therefore it is legitimate to *here* approximate $\sigma_x + \sigma_y$ in the form

$$\sigma_x + \sigma_y = 12\frac{M_x + M_y}{h^2}\frac{z}{h}. \tag{11}$$

It turns out that, numerically, the effect of the $(\sigma_x + \sigma_y)$-term in (10) will be negligibly small compared to the effect of the τ_{xz}-term, for normal ranges of values of $\nu_z G/E_z$. We will limit ourselves here to a consideration of the problem subject to this restriction and replace equation (10) by the simplified relation

$$u_{,z} = -W_{,x} + \frac{3V_x}{2hG}\left(1 - 4\frac{z^2}{h^2}\right), \tag{12}$$

again with a corresponding relation for $v_{,z}$.

Having (12) we conclude immediately that to the same degree of approximation

$$u = -zW_{,x} + \frac{3V_x}{2G}\left(\frac{z}{h} - \frac{4}{3}\frac{z^3}{h^3}\right) \tag{13a}$$

together with

$$v = -zW_{,y} + \frac{3V_y}{2G}\left(\frac{z}{h} - \frac{4}{3}\frac{z^3}{h^3}\right). \tag{13b}$$

Introduction of (13) into equations (8), and observation of the equilibrium equation $V_{x,x} + V_{y,y} = 0$, gives as new supplementary information concerning the three-dimensional aspects of the problem of plate bending expressions for bending and twisting stresses of the form

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left[W_{,xx} + \nu W_{,yy} - 3\frac{1-\nu}{2Gh}V_{x,x}\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)\right] \tag{14a}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left[W_{,yy} + \nu W_{,xx} - 3\frac{1-\nu}{2Gh}V_{y,y}\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)\right] \tag{14b}$$

$$\tau_{xy} = -\frac{Ez}{1+\nu}\left[W_{,xy} - \frac{3}{4Gh}(V_{x,y} + V_{y,x})\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)\right]. \tag{14c}$$

Having equations (14) we obtain expressions for bending and twisting couples in the usual way, with $D = Eh^3/12(1-\nu^2)$, in the form

$$M_x = -D\left[W_{,xx} + \nu W_{,yy} - \frac{6}{5}\frac{1-\nu}{Gh}V_{x,x}\right] \tag{15a}$$

$$M_y = -D\left[W_{,yy} + \nu W_{,xx} - \frac{6}{5}\frac{1-\nu}{Gh}V_{y,y}\right] \tag{15b}$$

$$H_{xy} = -(1-\nu)D\left[W_{,xy} - \frac{3}{5}\frac{1}{Gh}(V_{x,y} + V_{y,x})\right] \tag{15c}$$

and we note that these equations are in *exact* agreement with the corresponding relations (II) to (IV) in [2], upon introducing the stipulation that $p = 0$, and upon setting $E = 2(1+\nu)G$ in equations (15).†

For what follows it is of importance to compare the contents of (15) with the expressions for the surface values of the stresses σ_x, σ_y, τ_{xy} as given by equations (14). Evidently, we have from equations (14),

$$\sigma_x\left(\pm\frac{h}{2}\right) = \mp\frac{1}{2}\frac{Eh}{1-\nu^2}\left[W_{,xx} + \nu W_{,yy} - \frac{1-\nu}{Gh}V_{x,x}\right] \tag{16a}$$

†We note in passing the possibility of introducing effective *rotational* displacement components φ_x and φ_y, so that

$$V_x = \frac{5Gh}{6}(\varphi_x + W_{,x}), \quad V_y = \frac{5Gh}{6}(\varphi_y + W_{,y})$$

and therewith, in place of equations (15),

$$M_x = D(\varphi_{x,x} + \nu\varphi_{y,y}), \quad M_y = D(\varphi_{y,y} + \nu\varphi_{x,x}), \quad 2H_{xy} = (1-\nu)D(\varphi_{x,y} + \varphi_{y,x}).$$

$$\sigma_y\left(\pm\frac{h}{2}\right) = \mp\frac{1}{2}\frac{Eh}{1-\nu^2}\left[W_{,yy} + \nu W_{,xx} - \frac{1-\nu}{Gh}V_{y,y}\right] \tag{16b}$$

$$\tau_{xy}\left(\pm\frac{h}{2}\right) = \mp\frac{1}{2}\frac{Eh}{1+\nu}\left[W_{,xy} - \frac{1}{2Gh}(V_{x,y} + V_{y,x})\right] \tag{16c}$$

and the noteworthy difference between the right hand sides of (15) and (16) is the absence in equations (16) of the factor 6/5 which occurs in front of the terms with V_x and V_y in equations (15). To show the significance of this difference we consider in what follows its effect on the solution of the problem of the stress concentration due to the presence of a circular hole in the otherwise uniform transverse pure bending of an infinite plate, as previously considered in [2].

Introducing polar coordinates r, θ and making use of the notation and also of the results in [2] the problem which is now being considered is defined by means of the boundary conditions

$$r = \infty: \quad M_r = M_0\frac{1+\cos 2\theta}{2}, \quad H_{r\theta} = M_0\frac{\sin 2\theta}{2}, \quad V_r = 0 \tag{17a}$$

$$r = a: \quad M_r = 0, \quad H_{r\theta} = 0, \quad V_r = 0. \tag{17b}$$

To be determined on the basis of these boundary conditions are in particular a stress couple concentration factor k_B, defined by $k_B = M_\theta(a, \pi/2)/M_0$, and in addition to this a maximum-stress concentration factor k_B^*, defined by $k_B^* = \sigma_\theta(a, \pi/2, h/2)/\sigma_0$, where $\sigma_0 = 6M_0/h^2$.

In order to evaluate k_B and k_B^* we make use of expressions for M_θ and $\sigma_\theta(h/2)$ which in accordance with equations (15) and (16), in association with equation (IIIa) in [2], are

$$M_\theta = -D\left[\frac{W_{,r}}{r} + \frac{W_{,\theta\theta}}{r^2} + \nu W_{,rr} - \frac{6}{5}\frac{1-\nu}{Gh}\left(\frac{V_{\theta,\theta}}{r} + \frac{V_r}{r}\right)\right] \tag{18}$$

$$\sigma_\theta\left(\frac{h}{2}\right) = -\frac{6D}{h^2}\left[\frac{W_{,r}}{r} + \frac{W_{,\theta\theta}}{r^2} + \nu W_{,rr} - \frac{1-\nu}{Gh}\left(\frac{V_{\theta,\theta}}{r} + \frac{V_r}{r}\right)\right]. \tag{19}$$

In this we now make use of the expressions for W, V_θ and V_r which were obtained in [2] for the boundary value problem defined by the boundary conditions (17) and by the associated two-dimensional differential equations. We furthermore make use of various relations and determinations carried out in [2]. These leave us with expressions for $M_\theta(a, \theta)$ and $\sigma_\theta(a, \theta, h/2)$ which are of the following form

$$M_\theta(a, \theta) = M_0\left\{1 - \cos 2\theta\frac{2(1+\nu)K_2(\mu)}{(1+\nu)K_2(\mu) + 2K_0(\mu)}\right\}, \tag{20}$$

and

$$\sigma_\theta\left(a, \theta, \frac{h}{2}\right) = \sigma_0\left\{1 - \cos 2\theta\frac{2(1+\nu)K_2(\mu) - 4K_1(\mu)/3\mu}{(1+\nu)K_2(\mu) + 2K_0(\mu)}\right\}. \tag{21}$$

In this K_0, K_1 and K_2 are modified Bessel functions and $\mu = \sqrt{10}a/h$.

Equations (20) and (21) imply as expressions for k_B and k_B^*

$$k_B = \frac{3(1+\nu)K_2 + 2K_0}{(1+\nu)K_2 + 2K_0}, \quad k_B^* = \frac{3(1+\nu)K_2 + 2K_0 - 4K_1/3\mu}{(1+\nu)K_2 + 2K_0}. \tag{22}$$

Having equations (22) we are now in a position to compare our results with the results of an exact analysis of this stress concentration problem, within the framework of three-dimensional elasticity theory, by Alblas [1]. It was found by Alblas that his exact values for the stress couple concentration factor k_B were in close agreement with the corresponding approximate values in [2]. Alblas also obtained what amounted to exact values for the maximum-stress concentration factor k_B^* and he showed that, in the range of values of the ratio $2a/h$ which he considered k_B^* may differ from k_B by as much as 10 per cent ([1], p. 99), consistent with the observation in [2] that "if standard plate theory gives a value of 1·5 and the present theory a value 2·0 (for the factor of concentration of stress as approximated by the stress *couple* concentration factor) then it is believed that the actual value lies in between 1·90 and 2·10". In this context it is of special significance that the new approximate maximum-stress concentration factor k_B^*, in equation (22), does in fact give numerical values which are in excellent agreement with Alblas' values in the $2a/h$-range considered by him, with the differences between Alblas' exact values of k_B^* and the present approximate values being considerably smaller than 1 per cent, in accordance with the data in Table 1.

Table 1. Values of stress couple concentration factors k_B and of maximum-stress concentration factors k_B^* for transverse pure bending of an infinite isotropic homogeneous plate with a circular hole, as a function of the diameter-thickness ratio $2a/h$, with Poisson's ratio $\nu = 1/4$

	k_B		k_B^*	
$\frac{2a}{h}$	*A*	*R*	*A*	*R*
0	—	3	—	2·47
0·5	—	2·50	—	2·18
1	2·268	2·243	2·052	2·038
2	2·045	2·038	1·938	1·922
3	1·960	1·956	1·865	1·875
4	1·914	1·912	1·841	1·850
5	1·896	1·885	1·830	1·835
∞	1·769	1·769	1·769	1·769

REFERENCES

1. J. B. Alblas, Theorie van de Driedimensionale Spanningstoestand in een Doorboorde Plaat. Dissertation, Delft (1957).
2. E. Reissner, The effect of transverse shear deformation on the bending of elastic plates. *J. Appl. Mech.* **12**, A-69-A-77 (1945).